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Computation of Superpotentials for D-Branes

Paul S. Aspinwall

Department of Physics and SLAC
Stanford University
Stanford, CA 94305/94309
and
Center for Geometry and Theoretical Physics
Box 90318
Duke University
Durham, NC 27708-0318

Sheldon Katz

Departments of Mathematics and Physics
University of Illinois at Urbana-Champaign
Urbana, IL 61801

Abstract

We present a general method for the computation of tree-level superpotentials for the world-volume theory of B-type D-branes. This includes quiver gauge theories in the case that the D-brane is marginally stable. The technique involves analyzing the A_∞-structure inherent in the derived category of coherent sheaves. This effectively gives a practical method of computing correlation functions in holomorphic Chern–Simons theory. As an example, we give a more rigorous proof of previous results concerning 3-branes on certain singularities including conifolds. We also provide a new example.

1 Introduction

Consider a type II superstring compactification on a Calabi–Yau threefold X. BPS D-branes that "wrap cycles" within X and fill the noncompact spacetime give rise to an effective $\mathcal{N} = 1$, $d = 4$ supersymmetric gauge theory arising from the D-brane world-volume. As is well-known [1], if N irreducible D-branes wrap the same cycle, one obtains a model with $\mathrm{U}(N)$ gauge symmetry.

One may obtain more general supersymmetric gauge theories in the form of quiver gauge theories by considering marginally stable D-branes. Suppose a given D-brane (which may consist of multiple copies of some irreducible D-branes) is marginally stable with respect to decay into N_1 copies of some (irreducible) D-brane plus N_2 copies of another D-brane, etc., then one obtains a gauge theory with gauge group $\mathrm{U}(N_1) \times \mathrm{U}(N_2) \times \ldots$. The fact that the given D-brane is marginally stable means that there will be massless open strings between the decay products. These strings give rise to massless chiral supermultiplets in bifundamental $(\overline{\mathbf{N}}_1, \mathbf{N_2})$ representations etc.

These $\mathcal{N} = 1$, $d = 4$ supersymmetric gauge theories will, in general, have a nontrivial superpotential expressible as a function of the chiral superfields. The purpose of this paper is to describe a systematic and general method for the computation of this superpotential at tree level directly from the algebraic geometry of X.

There are two types of BPS D-branes on a Calabi–Yau threefold — the so-called A-type and B-type. The A-type D-branes are described by special Lagrangian cycles within X [2] and are, in principle, described completely by the language of the Fukaya category [3–5]. Having said that, the Fukaya category is extremely difficult to deal with explicitly for any example of a Calabi–Yau threefold.

The other "B-type" D-branes are described by $\mathbf{D}(X)$, the derived category of coherent sheaves on X [6–9]. While, at first sight, the derived category may appear to be mathematically formidable, it is actually very useful for direct computations in any given example.

In this paper we therefore focus on the problem of computing the superpotential for B-type D-branes. The easiest route for computing superpotentials for A-branes in many situations is by reducing to the B-brane case by using mirror symmetry.

Various methods leading to proposals for superpotentials in several examples have already appeared in the literature [10–16]. Each of these papers has used a somewhat indirect approach to analyzing the superpotential. Here we will give a very direct method that can be applied for any collection of B-type D-branes on any Calabi–Yau manifold. It uses similar ideas to those used in demonstrations of homological mirror symmetry given in [17].

In a recent paper [18] this problem was studied in the Landau–Ginzburg phase of the B-model. It is believed that this should give the same result as in the Calabi–Yau phase, i.e., the case of interest here. The only worry would be that, in current understanding, the Landau–Ginzburg analysis appears to collapse the derived category somewhat by identifying objects under a shift of 2. This might change the results of computations of superpotentials in some cases.

It has long been known [19] that the information concerning the superpotential is contained in a holomorphic Chern–Simons theory of the Calabi–Yau threefold in question. The

propagator in this field theory appears to require a complete knowledge of the metric of X and so cannot be computed in general. Here we recast the holomorphic Chern–Simons theory in a form described purely by homological algebra and algebraic geometry.

The logic of this argument is very similar to that used when arguing that B-branes, which are originally described by Dolbeault cohomology, are ultimately described by $\mathbf{D}(X)$. Indeed, all we need do is to supplement this argument by a product structure. The wedge product of Dolbeault cohomology becomes a "composition" product in $\mathbf{D}(X)$.

Having done this change of language to algebraic geometry one can then use Čech cohomology and a knowledge of locally-free resolutions to perform a computation of the superpotential. Our method applies, in principle, to a computation involving any D-brane (i.e., any object in the derived category) on any Calabi–Yau threefold. The only obstacle in general to the computation is the stamina required to compute Čech cohomology when several patches are required, and dealing with potentially long locally-free resolutions.

All the technical machinery of computing superpotentials is tied up in the A_∞-algebra language (see, for example, [20, 21]). We therefore begin with a review of the required facts in section 2.

In section 3 we discuss general features of the way that the superpotential is described by correlation functions in the topological B-model. An interesting result is that, although one can define a generalized superpotential on the "thickened" moduli space of the topological field theory, it essentially contains no more information than the physical superpotential. We also discuss the uniqueness of the superpotential computed by the topological field theory.

In section 4 we show how holomorphic Chern–Simons theory can be restated in terms more appropriate to algebraic geometry and in section 5 we compute some examples. We are able to verify some results concerning 3-branes on conifold singularities. We also compute a new result based on a 5-brane wrapping a particular $\mathbb{P}^1$.

2 A_∞ Algebras and Categories

The superpotential in these $\mathcal{N} = 1$ gauge theories is intimately related to the structure of an A_∞ algebra (in the case of a single stable D-brane) or category (in the case of a quiver). This has been discussed in [14, 22–24]. We begin, therefore, with a review of A_∞ algebras following [20, 21].

Let V be a vector space with a $\mathbb{Z}$-grading and let $T(V)$ be the resulting graded tensor algebra

$$T(V) = \bigoplus_{n=1}^{\infty} V^{\otimes n}. \tag{1}$$

If $a \in V$, we will denote the grade of a by $|a|$. By the usual abuse of notation we will often write $(-1)^a$ rather than $(-1)^{|a|}$. If f and g are operators of given degrees, we use the rule

$$(f \otimes g)(a \otimes b) = (-1)^{|g|.|a|} f(a) \otimes g(b). \tag{2}$$

Now let $\mathbf{d}$ be a derivative with degree 1, with respect to the grading, acting on $T(V)$ obeying the graded Leibniz rule

$$\mathbf{d}(a \otimes b) = \mathbf{d}(a) \otimes b + (-1)^a a \otimes \mathbf{d}(b). \tag{3}$$

We also demand

$$\mathbf{d}^2 = 0. \tag{4}$$

The Leibniz rule (3) means that $\mathbf{d}$ is entirely determined by its restriction to V. Let us denote this restriction as $(\mathbf{d})_V$. One can then decompose

$$(\mathbf{d})_V = d_1 + d_2 + \ldots, \tag{5}$$

where

$$d_k : V \to V^{\otimes k}. \tag{6}$$

Let $V[1]$ denote the vector space V with all the grades decreased by one and let $s : V \to V[1]$ be the obvious map of degree -1. We can now define our A_∞ algebra A:

$$A = (V[1])^*. \tag{7}$$

together with its higher products

$$m_k : A^{\otimes k} \to A, \tag{8}$$

given by the dual of $s^{\otimes k} \cdot d_k \cdot s^{-1}$. The map m_k thus has degree $2 - k$.[1]

The condition (4) then becomes equivalent to [25]

$$\sum_{r+s+t=n} (-1)^{r+st} m_u(\mathbf{1}^{\otimes r} \otimes m_s \otimes \mathbf{1}^{\otimes t}) = 0, \tag{9}$$

for any $n > 0$, where $u = n + 1 - s$. One may view (9) as the defining relations for an A_∞ algebra.

It is easy to extend the idea of an A_∞-algebra to an A_∞-category [26]. Such a category consists of objects and morphisms in the usual way except that morphisms, under k-fold compositions, satisfy the relations (9). In particular, an A_∞-category need not be a category in the usual sense since composition of morphisms need not be associative.

Now suppose we have another graded vector space U with its own differential $\mathbf{d}$ acting on $T(U)$. It is natural to consider maps $g : T(U) \to T(V)$ which commute with $\mathbf{d}$. We impose the condition

$$g(a \otimes b) = (-1)^{|g||a|} g(a) \otimes g(b), \tag{10}$$

so that such maps are defined completely by their restriction to U.

[1]The grades of vector spaces are negated upon dualizing. Thus, when a map between vector spaces is dualized, its direction is reversed but its degree remains the same.

If B is the A_∞-algebra constructed from U, such a g gives rise to an "A_∞-morphism" given by maps

$$f_k : A^{\otimes k} \to B, \tag{11}$$

constructed in the obvious way from g above. The condition that g commutes with $\mathbf{d}$ then becomes

$$\sum_{r+s+t=n} (-1)^{r+st} f_u(\mathbf{1}^{\otimes r} \otimes m_s \otimes \mathbf{1}^{\otimes t}) = \sum_{\substack{1\leq r\leq n \\ i_1+\ldots+i_r=n}} (-1)^q m_r(f_{i_1} \otimes f_{i_2} \otimes \cdots \otimes f_{i_r}), \tag{12}$$

for any $n > 0$ and $u = n + 1 - s$ again. The sign on the right is given by

$$q = (r-1)(i_1-1) + (r-2)(i_2-1) + \ldots + (i_{r-1}-1). \tag{13}$$

Note that $m_1 : A \to A$ is a degree one map satisfying $m_1 \cdot m_1 = 0$. It thus gives A the structure of a graded differential complex, and we may take cohomology to yield $H^*(A)$. By choosing representatives of each cohomology class we may define an embedding

$$i : H^*(A) \hookrightarrow A. \tag{14}$$

Thanks to a theorem by Kadeishvili [27], we may define an A_∞ structure on $H^*(A)$ such that

1. There is an A_∞ morphism f from $H^*(A)$ to A with f_1 equal to the embedding i.
2. $m_1 = 0$.

Here, m_1 refers to the A_∞ structure on $H^*(A)$. This A_∞ structure is not unique, but it is unique up to A_∞-isomorphisms, as will be discussed in a more general situation in Lemma 1 below. An A_∞-algebra with $m_1 = 0$ is called a *minimal* A_∞-algebra.

It is quite easy to construct Kadeishvili's A_∞-structure in practice. A rather simple example of an A_∞-algebra is given by $m_k = 0$ for $k \geq 3$. Such an algebra is called a *differential graded algebra*, or dga. In this paper, we will need to put an A_∞ structure on the cohomology of a dga, which may be done explicitly as follows. Let m_1 on A be denoted d, and let $m_2(a \otimes b)$ be denoted $a \cdot b$.

Putting $n = 2$ in (12), and using the fact that $m_1 = 0$ in $H^*(A)$, yields

$$im_2 = (i \cdot i) + df_2. \tag{15}$$

Since $d(i \cdot i) = 0$, we must define m_2 on $H^*(A)$ as the cohomology class of $i \cdot i$. We may also use this to define a choice of $f_2 : H^*(A)^{\otimes 2} \to A$ (up to an element in the kernel of d). Next, putting $n = 3$ in (12) yields[2]

$$im_3 = f_2(\mathbf{1} \otimes m_2) - f_2(m_2 \otimes \mathbf{1}) + (i \cdot f_2) - (f_2 \cdot i) + df_3. \tag{16}$$

[2] There appears to be a typo in [25] discarding too many terms.

Direct computation using (15) shows that $d(f_2(\mathbf{1}\otimes m_2)-f_2(m_2\otimes\mathbf{1})+(i\cdot f_2)-(f_2\cdot i))=0$, so as before this defines m_3 and allows us to choose a definition for f_3. Clearly this process continues and defines all the products for the A_∞ algebra on $H^*(A)$.

This construction of the A_∞ algebra on $H^*(A)$ may be rephrased following [26, 28] in a language which will also be useful to us. Suppose we define a projection $p : A \to H^*(A)$ such that $p\circ i = 1$ and furthermore assume that we have a map $H : A \to A$ of degree -1 such that $1 - i\circ p = dH + Hd$.

Clearly, m_2 is defined as $p\circ(i\cdot i)$ as before. For $k>2$, we then define

$$m_k = \sum_T \pm m_{k,T}, \tag{17}$$

where the sum is over all trees T with k branch tips at the top and one root. These trees look like Feynman diagrams of a ϕ^3 field theory and are computed accordingly, with m_2 of A acting as the cubic coupling and H acting as the propagator. We refer to [26] for more details.

We say that two dga's A and B are *quasi-isomorphic* if there is a homomorphism of dga's $g : A \to B$ (i.e. preserving the respective products and commuting with the differentials) inducing an isomorphism on the respective cohomologies, which can then be identified. A simple extension of the "uniqueness up to A_∞-isomorphism" part of the above construction proves the following.

Lemma 1 *Suppose that A and B are quasi-isomorphic dga's, determining A_∞ structures on $H^*(A)\simeq H^*(B)$ as above. Then these two A_∞ algebras are A_∞-isomorphic.*

Before describing the simple proof, we recall that A_∞-quasi-isomorphisms are simply A_∞-morphisms which are quasi-isomorphisms, i.e. which induce isomorphisms between the respective m_1-cohomologies. We will need the result that A_∞-quasi-isomorphisms have homotopy inverses, see [25] and the references therein. We also need the result that an A_∞-morphism f between minimal A_∞-algebras is an isomorphism if and only if f_1 is an isomorphism.

Let f be an A_∞-quasi-isomorphism from $H^*(A)$ to A and g be an A_∞-quasi-isomorphism from $H^*(B)$ to B as in Kadeishvili's theorem. Let $\phi : A \to B$ be the given quasi-isomorphism of dga's, which, viewing A and B as A_∞-algebras can be viewed as describing an A_∞-quasi-isomorphism. Let h be a homotopy inverse of g. Then $r = h\circ\phi\circ f$ is an A_∞-morphism from $H^*(A)$ to $H^*(B)$. Here $\circ$ denotes the composition of A_∞-morphisms, see e.g. [25]. But $r_1 : H^*(A)\to H^*(B)$ is an isomorphism by definition since ϕ is a quasi-isomorphism, while $H^*(A)$ and $H^*(B)$ are minimal A_∞-algebras. Hence r is an isomorphism by the discussion above.

3 D-Branes and Superpotentials

To begin with, assume we have a D-brane that consists of a vector bundle $E \to X$. This was the case studied by Witten in [19]. The open strings in the B-model correspond to elements

of Dolbeault cohomology $H^{0,q}_{\bar\partial}(X, \mathrm{End}(E))$.

The vertex operators corresponding to $q = 0$ yield massless vector bosons in the uncompactified 4 dimensions which give rise to a gauge theory. The vector bundle E is said to be *simple* if $\mathrm{Hom}(E, E) = \mathrm{End}(E) = \mathbb{C}$. In this case we have one vector boson and the gauge group is U(1). Similarly if E is $(E_0)^{\oplus N}$, where E_0 is simple, then the gauge group is $\mathrm{U}(N)$.

The vertex operators corresponding to $q = 1$ yield massless scalars and fermions in four dimensions coming from chiral supermultiplets. These transform in the adjoint representation of $\mathrm{U}(N)$. Let A denote the Hilbert space of open string states. The effective D-brane world-volume theory contains a superpotential which is a holomorphic function of these chiral superfields.

In [29] it was shown that this superpotential could be computed in terms of correlation functions in the associated *topological* quantum field theory. The result is as follows.

The open strings are associated to local vertex operators ψ_i in the topological field theory. These ψ_i's may be viewed as a basis for A. To each such vertex operator, one may construct a 1-form operator

$$\psi_i^{(1)} = \tfrac{1}{\sqrt{2}}\left\{G^-_{-\frac{1}{2}} + \overline{G}^-_{-\frac{1}{2}}, \psi_i\right\}, \tag{18}$$

These 1-form operators may be used to deform the topological field theory (at least to first order):

$$S \to S + \sum_i Z_i \psi_i^{(1)}, \tag{19}$$

where the Z_i are complex numbers as far as the topological field theory is concerned. The Z_i are (the scalar components of) chiral superfields in the effective world-volume theory. The deformations (19) correspond to giving vacuum expectations values to these fields. Thus, *the chiral superfields are naturally dual to the vertex operators of the topological quantum field theory.*

Let $q_i \in \mathbb{Z}$ denote the ghost number of ψ_i. Then $\psi_i^{(1)}$ has ghost number $q_i - 1$. The only operators that can be used to deform the untwisted conformal field theory associated to the D-brane must have $q_i = 1$. We would like to extend our discussion to the "thickened" moduli space of [30]. In this picture, all ghost numbers are allowed. This gives rise to a generalized space of chiral superfields where Z_i has a grade $1 - q_i$. We also have a generalized superpotential $\mathbf{W}$ which is a function of all the Z_i's. Always remember, though, that only the fields of grade zero are true chiral superfields.

Following the conventions of [24], we define correlation functions for $k + 1$ open string vertex operators:

$$B_{i_0,i_1,\ldots,i_k} = (-1)^{\zeta_1+\zeta_2+\ldots+\zeta_{k-1}} \langle \psi_{i_0}\, \psi_{i_1}\, P \int \psi^{(1)}_{i_2} \int \psi^{(1)}_{i_3} \ldots \int \psi^{(1)}_{i_{k-1}}\, \psi_{i_k}\rangle, \tag{20}$$

where we introduce the notation $\zeta_j = 1 - q_{i_j}$. The integrals in this correlation function are over segments of the boundary so as to preserve the path ordering. A choice of regulator needs to be made in order to fully define these correlation functions as was done in [24]. We will avoid making such a choice, giving rise to ambiguities which we discuss at the end of this section.

It was shown in [24] that these correlators satisfy the following cyclicity property

$$B_{i_0,i_1,\ldots,i_k} = (-1)^{\zeta_k(\zeta_0+\zeta_1+\ldots+\zeta_{k-1})} B_{i_k,i_0,i_1,\ldots,i_{k-1}}. \tag{21}$$

In the case of N copies of a simple D-brane, the fields Z_i naturally form $N \times N$ matrices. We may now write the superpotential

$$\mathbf{W} = \operatorname{Tr}\left(\sum_{k=2}^{\infty} \sum_{i_0,i_1,\ldots,i_k} \frac{B_{i_0,i_1,\ldots,i_k}}{k+1} Z_{i_0} Z_{i_1} \ldots Z_{i_k} \right). \tag{22}$$

Note that this trace has a graded cyclicity property consistent with (21).

The correlation functions of a topological quantum field theory are subject to various constraints due to sewing conditions as discussed in [31,32] and, in particular, [33]. The open string "pair of pants" diagram associates a bilinear product of degree 0 to A. Anticipating the connection with A_∞ algebras, we denote this

$$m_2 : A \otimes A \to A. \tag{23}$$

If X is a Calabi–Yau threefold, there is also a "trace map" of degree -3

$$\gamma : A \to \mathbb{C}. \tag{24}$$

It follows that our desired correlation function may be written in the form

$$B_{i_0,i_1,\ldots,i_k} = \gamma\left(m_2\left(m_k(\psi_{i_0}, \psi_{i_1}, \ldots, \psi_{i_{k-1}}), \psi_{i_k} \right) \right), \tag{25}$$

for maps of degree $2-k$

$$m_k : A^{\otimes k} \to A. \tag{26}$$

It was shown in [24] that these products do indeed obey the conditions (9) and thus give A the structure of an A_∞ algebra.

Comparing this structure with the description of A_∞ algebras in section 2, it should be clear that the chiral superfields Z_i play the role of generators of the space V. The shift by one comes from (18) and the dualizing comes from (19). Since the structure of the A_∞ algebra is simpler to describe in terms of $T(V)$, it should be enlightening to rephrase the above in this language.

The degree -3 pairing $\gamma(m_2(-,-))$ is non-degenerate on A and simply corresponds to Serre duality. It naturally dualizes to produce a map

$$\eta : \mathbb{C} \to V \otimes V, \tag{27}$$

of degree -1.

If Z_i is a homogeneous basis for V,

$$\eta(1) = \sum_i Z_i \otimes \hat{Z}_i, \tag{28}$$

where $\hat{Z}_i$ are viewed as the "Serre dual" of Z_i.

We write a basis of V as follows. Let $X_1 \dots X_n$ be a basis of the degree 0 part of V. These are therefore the true chiral superfields in the four-dimensional theory. Assume, for now, that E is simple, i.e., $\mathrm{Hom}(E, E) = \mathbb{C}$. In other words, there is a unique "identity" vertex operator for open strings beginning and ending on E. This is dual to an element denoted $e \in V$ of degree 1. Serre duality can now be used to give a basis $\hat{X}_i$ of the degree -1 part of V and a generator $\hat{e}$ of the degree -2 part of V, where

$$\eta(1) = e \otimes \hat{e} + \hat{e} \otimes e + \sum_\alpha X_\alpha \otimes \hat{X}_\alpha + \sum_\alpha X_\alpha \otimes \hat{X}_\alpha. \tag{29}$$

Viewing the superpotential $\mathbf{W}$ as an element of $T(V)$ and using (22), the higher products of the A_∞ algebra can be rephrased in the language of section 2 as the beautifully simple statement

$$\mathbf{d}\hat{Z}_i = \frac{\partial \mathbf{W}}{\partial Z_i}. \tag{30}$$

Since $T(V)$ is the *non-commutative* algebra generated by Z_i, some care is needed in defining the partial derivative in (30). The recipe is as follows. The cyclic trace property (21) allows $\mathbf{W}$ to be written with any of the generators at the front. $\partial \mathbf{W}/\partial Z_i$ is then defined as the sum of all the possible forms of W under the trace property with Z_i at the front, with said Z_i removed. Clearly this coincides with the usual definition of derivative in commutative algebra.

The identity vertex operator has special properties under the higher products as shown in [24]. Let ψ_0 be the identity operator. Then

$$\begin{aligned} m_2(\psi_0, \psi_i) = m_2(\psi_i, \psi_0) &= \psi_i \\ m_k(\psi_{i_1}, \psi_{i_2}, \dots, \psi_0, \dots) &= 0, \qquad \text{for } k > 2. \end{aligned} \tag{31}$$

Carefully computing signs, it follows from (30) that

$$\mathbf{W} = \frac{1}{2}\sum_i \Big((-1)^{Z_i} \hat{Z}_i \otimes e \otimes Z_i - \hat{Z}_i \otimes Z_i \otimes e \Big) + \text{terms not containing } e. \tag{32}$$

In (32) we have also dropped terms which can be deduced from the cyclicity property (21).

So far we have discussed one simple D-brane. It is very easy to generalize to the case of a collection of D-branes

$$E_1^{\oplus N_1} \oplus E_2^{\oplus N_2} \oplus \dots, \tag{33}$$

forming a $\mathrm{U}(N_1) \times \mathrm{U}(N_2) \times \dots$ quiver gauge theory, where each E_j is simple. In order to form a quiver gauge theory free from tachyons or peculiar vector bosons we are required to impose [34, 35]

$$\mathrm{Hom}(E_j, E_k) = 0, \qquad \text{for } j \neq k. \tag{34}$$

This means that the only degree zero vertex operators in A remain multiples of identity maps $E_j \to E_j$.

The effect of passing to a quiver gauge theory is that we must now think in terms of A_∞ categories rather than algebras. This amounts to little more than bookkeeping as follows. The elements of A should be viewed as morphisms between D-branes and, as such, as elements of $H^{0,*}_{\bar\partial}(X, \mathrm{Hom}(E_i, E_j))$. All we need do is to rewrite (32) as

$$\mathbf{W} = \frac{1}{2}\,\mathrm{Tr}\left(\sum_i (-1)^{Z_i}\hat{Z}_i \otimes e \otimes Z_i - \sum_i \hat{Z}_i \otimes Z_i \otimes e + \text{terms not containing } e\right), \tag{35}$$

where now Z_i are matrices. The "$\otimes$" in (35) now implicitly includes matrix multiplication and the concept of composition of morphisms between different objects. The symbol e now refers to a square $N_j \times N_j$ matrix with entries dual to the identity operator of a given simple D-brane E_j. The composition of morphisms implied by the superpotential must begin and end on the same D-brane so that a trace may then be taken. This is equivalent to the statement that the superpotential is gauge invariant.

The equation (30) remains valid for the quiver theory. One implicitly removes the trace and then naïvely applies the rule for differentiation we described above.

The general form of the superpotential can be further constrained. The only vertex operators appearing have grade 0, 1, 2 or 3. Thus, the Z_i have grade 1, 0, -1 or -2, with the e the only generator with grade 1. Now, $\mathbf{d}e$ is of degree 2 and so must be a sum of terms in $T(V)$ each with at least one e. The property of the identity element of an A_∞ algebra thus implies

$$\mathbf{d}e = -e \otimes e. \tag{36}$$

Similarly, since X_α has degree 0, we must have

$$\mathbf{d}X_\alpha = X_\alpha \otimes e - e \otimes X_\alpha. \tag{37}$$

Since $\hat{X}_\alpha$ is of degree -1,

$$\mathbf{d}\hat{X}_\alpha = F(X_\beta) - \hat{X}_\alpha \otimes e - e \otimes \hat{X}_\alpha, \tag{38}$$

where $F(X_\beta)$ is an arbitrary function of the X_β's. Finally $d\hat{e} = \partial\mathbf{W}/\partial e$ and is completely determined by (35). The result is that

$$\mathbf{W} = \mathrm{Tr}\left(W(X_\alpha) - \hat{e}\otimes e \otimes e + \sum_\alpha \left(\hat{X}_\alpha \otimes X_\alpha \otimes e - \hat{X}_\alpha \otimes e \otimes X_\alpha\right)\right), \tag{39}$$

where $W(X_\alpha)$ is a completely arbitrary function of all the chiral superfields X_α. This function is, of course, the physical superpotential.

For any $Z_i \in V$, using (30) and (39) it is a simple matter to show

$$\mathbf{d}^2 Z_i = 0. \tag{40}$$

There are therefore three remarkable properties of (39):

1. The A_∞ relations are trivially satisfied. There is no need to go through the computation of [24].

2. The generalized superpotential associated to the thickened moduli space is determined completely by $W(X_\alpha)$ — the physical superpotential on the physical moduli space.

3. The A_∞ relations are satisfied for completely arbitrary $W(X_\alpha)$.

We should perhaps point out that much of the simplification we have found here is due to the constraint (34) for a physical quiver. Had we not imposed this, we could not have used the special properties of the identity operator.

The A_∞-morphisms are also simplified when passing to the dual language of the chiral superfields. An A_∞-morphism from a theory with superfields Z_i to a theory with superfields Y_α is simply an analytic map

$$Y_\alpha = g_\alpha(Z_1, Z_2, \ldots). \tag{41}$$

The complicated expression (12) is restated as g commuting with $\mathbf{d}$. If any f_k is nonzero for $k \geq 2$, this map of superfields is nonlinear.

We will be using Kadeishvili's theorem of section 2 to compute the desired A_∞-structure yielding the superpotential, It is important to note that this theory only gives this structure *up to an A_∞-isomorphism.* It follows that we will only be able to determine the superpotential *up to a nonlinear change in superfields* where this nonlinear map is invertible and commutes with $\mathbf{d}$.

It is not surprising that there is an ambiguity in the superpotential. From the four-dimensional field theory point of view, the topological B-model knows nothing about the kinetic term and so one is free to apply nonlinear redefinitions to the chiral superfields. From the point of the view of the string worldsheet, contact terms arise from the vertex insertion point coalescing at the ends of the integration regions. Such contact terms are known to introduce ambiguities as in [36].

That these ambiguities exist is therefore not a surprise, but we have a very precise form of the ambiguity — the nonlinear redefinition of the superfields must commute with $\mathbf{d}$. It would be interesting to find the physics behind this statement but we will not attempt to pursue this question here.

4 Holomorphic Chern–Simons Theory

In [19], it was shown how to exactly compute the correlation functions (20), at least for one D-brane E and for vertex operators in $H^{0,1}_{\bar\partial}(X, \mathrm{End}(E))$. One defines a *holomorphic Chern–Simons theory* with action

$$S = \int_X \mathrm{Tr}\left(\mathsf{A}\wedge\bar\partial\mathsf{A} + \tfrac{2}{3}\mathsf{A}\wedge\mathsf{A}\wedge\mathsf{A}\right)\wedge\Omega, \tag{42}$$

where the field A is a $(0,1)$-form on X taking values in $\mathrm{End}(E)$, and Ω is a holomorphic $(3,0)$-form on X.

From this, the correlation functions are then computed as follows in the language of section 3. Let A be the Hilbert space $H^{0,*}(X, \mathrm{End}(E))$. The trace map (24) is given by

$$\gamma(a) = \int_X \mathrm{Tr}(a) \wedge \Omega, \tag{43}$$

while

$$m_2(a,b) = a \wedge b, \tag{44}$$

where composition in $\mathrm{End}(E)$ is implicit. The computation of m_k is then exactly as described by the tree construction at the end of section 2. The propagator is, of course, the propagator of (42) which is given by $H = G\bar{\partial}^\dagger$, where G is the Green's operator inverting the Laplacian. Since, for any differential form α, [28]

$$\alpha = [\alpha]_{\mathrm{Harm}} + \bar{\partial}G\bar{\partial}^\dagger + G\bar{\partial}^\dagger\bar{\partial}, \tag{45}$$

we have the following (which was also effectively noted in [14]):

Theorem 1 *The correlation functions in the holomorphic Chern–Simons theory are associated with the A_∞ algebra as computed in section 2, where the dga is given by the Dolbeault complex of* $\mathrm{End}(E)$*-valued* $(0,q)$*-forms together with the wedge product. The embedding, i, of* $H^{0,*}(X, \mathrm{End}(E))$ *into this complex is given by Harmonic forms.*

This formulation of holomorphic Chern–Simons theory is all very well but it is not very practical. Computing the propagator $G\bar{\partial}^\dagger$ would appear to require a knowledge of the metric on X. Naturally this is not in the spirit of the topological field theory. One general expects all computations in the topological B-model to be cast in the language of algebraic geometry and thus not require detailed knowledge of X, such as its metric.

The derived category program for B-branes [7] precisely does this translation to algebraic geometry as reviewed in [9]. We need to extend this argument to include product structures. What we will arrive at is an A_∞-structure implicit in the derived category that has been discussed in [17, 37, 38]. Indeed, the equivalence we derive in this section was also described in these references.

The key idea is that we have three natural dga's associated to three different cohomologies, all of which may equally be used to analyze the problem at hand. Suppose we have a holomorphic vector bundle B with a product $\mu : B \otimes B \to B$. Let $\mathscr{B}$ be the locally-free sheaf of sections of B. In the case of interest, we will want $B = \mathrm{End}(E)$, and $\mathscr{B} = \mathscr{H}om(\mathscr{E}, \mathscr{E})$ with the product μ being given by composition. The useful dga's are then:

1. The Dolbeault complex of $(0,q)$-forms valued in B:

$$\ldots \xrightarrow{\bar{\partial}} \Gamma(\mathscr{A}^{0,q-1} \otimes \mathscr{B}) \xrightarrow{\bar{\partial}} \Gamma(\mathscr{A}^{0,q-1} \otimes \mathscr{B}) \xrightarrow{\bar{\partial}} \Gamma(\mathscr{A}^{0,q-1} \otimes \mathscr{B}) \xrightarrow{\bar{\partial}} \ldots, \tag{46}$$

where Γ denotes global section and $\mathscr{A}^{0,q}$ is the sheaf of C^∞ $(0,q)$-forms on X. This yields Dolbeault cohomology groups $H^*_{\bar{\partial}}(X, B)$. The product is given by the wedge product combined with μ. Putting $B = \mathrm{End}(E)$, this is the description Witten originally used to formulate the B-model [19].

2. The Čech complex of Čech cochains associated to an open cover $\mathfrak{U}$ for the locally-free sheaf $\mathscr{B}$ of sections of B:

$$\ldots \xrightarrow{\delta} \check{C}^{n-1}(\mathfrak{U},\mathscr{B}) \xrightarrow{\delta} \check{C}^{n}(\mathfrak{U},\mathscr{B}) \xrightarrow{\delta} \check{C}^{n+1}(\mathfrak{U},\mathscr{B}) \xrightarrow{\delta} \ldots \tag{47}$$

For sufficiently fine $\mathfrak{U}$, the cohomology of this complex yields the Čech cohomology groups $\check{H}^*(X,\mathscr{B})$. The product given by the cup product combined with μ yields the dga.

3. Given an injective resolution of $\mathscr{B}$:

$$0 \longrightarrow \mathscr{B} \longrightarrow \mathscr{I}^0 \xrightarrow{i_0} \mathscr{I}^1 \xrightarrow{i_1} \mathscr{I}^2 \xrightarrow{i_2} \ldots, \tag{48}$$

we may apply the global section functor, Γ, to yield a complex

$$\ldots \xrightarrow{\Gamma(i_{n-2})} \Gamma(\mathscr{I}^{n-1}) \xrightarrow{\Gamma(i_{n-1})} \Gamma(\mathscr{I}^{n}) \xrightarrow{\Gamma(i_{n})} \Gamma(\mathscr{I}^{n+1}) \xrightarrow{\Gamma(i_{n+1})} \ldots, \tag{49}$$

whose cohomology yields the sheaf cohomology groups $H^*(X,\mathscr{B})$. The resolution (48) extends μ naturally to a product:

$$\mu : \mathscr{I}^p \otimes \mathscr{I}^q \to \mathscr{I}^{p+q}, \tag{50}$$

which gives a dga structure to (49).

There is a standard spectral sequence argument, as reviewed in [9] which shows that these three theories of cohomology are equivalent. For example, one may define the double complex

$$E_0^{p,q} = \check{C}^p(\mathfrak{U}, \mathscr{B} \otimes \mathscr{A}^{0,q}). \tag{51}$$

To this we associate a single complex

$$E^n = \bigoplus_{p+q=n} E_0^{p,q}, \tag{52}$$

with differential $d = \delta + (-1)^p\bar{\partial}$. The d-cohomology of $E^\bullet$ can be realized as the abutment of either of two spectral sequences. The first spectral sequence has E_1-term obtained from the p-cohomology of (51):

$$E_1^{p,q} = \check{H}^p(\mathscr{B} \otimes \mathscr{A}^{0,q}), \tag{53}$$

and the second spectral sequence has E_1-term obtained from the q-cohomology of (51).

Since the $\mathscr{A}^{0,q}$ are fine sheaves (i.e. admit partitions of unity), so are the $\mathscr{B} \otimes \mathscr{A}^{0,q}$. It follows that their higher cohomologies vanish and (53) reduces to

$$E_1^{0,q} = \Gamma(\mathscr{B} \otimes \mathscr{A}^{0,q}) \tag{54}$$

with $E_1^{p,q} = 0$ for $p > 0$. Here $\Lambda^{0,q}$ is the ring of global $(0,q)$-forms on X. This spectral sequence therefore degenerates at E_2 and the d-cohomology of $E^\bullet$ is isomorphic to the cohomology of $E_1^{0,\bullet}$. In other words, the chain map

$$\begin{tikzcd} \cdots \arrow[r, "\bar\partial"] & \Gamma(\mathscr{B}\otimes\mathscr{A}^{0,n-1}) \arrow[r, "\bar\partial"] \arrow[d, "\xi"] & \Gamma(\mathscr{B}\otimes\mathscr{A}^{0,n}) \arrow[r, "\bar\partial"] \arrow[d, "\xi"] & \Gamma(\mathscr{B}\otimes\mathscr{A}^{0,n+1}) \arrow[r, "\bar\partial"] \arrow[d, "\xi"] & \cdots \\ \cdots \arrow[r, "d"] & E^{n-1} \arrow[r, "d"] & E^n \arrow[r, "d"] & E^{n+1} \arrow[r, "d"] & \cdots \end{tikzcd} \tag{55}$$

is a *quasi-isomorphism*. That is, ξ induces an isomorphism between the cohomology of the two complexes. Here, the ξ are the natural maps $\Gamma(\mathscr{B}\otimes\mathscr{A}^{0,k}) \to \check{C}^0(\mathfrak{U}, \mathscr{B}\otimes\mathscr{A}^{0,k}) \subset E^k$ expressing a global section in terms of the given open cover.

It is easy to see that ξ preserves the product structure between the two complexes too. Thus, ξ is a quasi-isomorphism of dga's.

Turning to the other spectral sequence, the fact that

$$0 \longrightarrow \mathscr{O} \xrightarrow{\varepsilon} \mathscr{A}^{0,0} \xrightarrow{\bar\partial} \mathscr{A}^{0,1} \xrightarrow{\bar\partial} \mathscr{A}^{0,2} \xrightarrow{\bar\partial} \ldots, \tag{56}$$

is exact (and remains exact upon tensoring with $\mathscr{B}$) for intersections in a suitably-chosen $\mathfrak{U}$ means that the q-cohomology of (51) is 0 unless $q = 0$, in which case we simply get $\check{C}^p(\mathfrak{U}, \mathscr{B})$. Thus this spectral sequence degenerates as well, and the cohomology of $\check{C}^\bullet(\mathfrak{U}, \mathscr{B})$ coincides with the d-cohomology of $E^\bullet$. More precisely, the chain map

$$\begin{tikzcd} \cdots \arrow[r, "\delta"] & \check{C}^{n-1}(\mathfrak{U},\mathscr{B}) \arrow[r, "\delta"] \arrow[d, "\varepsilon"] & \check{C}^{n}(\mathfrak{U},\mathscr{B}) \arrow[r, "\delta"] \arrow[d, "\varepsilon"] & \check{C}^{n+1}(\mathfrak{U},\mathscr{B}) \arrow[r, "\delta"] \arrow[d, "\varepsilon"] & \cdots \\ \cdots \arrow[r, "d"] & E^{n-1} \arrow[r, "d"] & E^n \arrow[r, "d"] & E^{n+1} \arrow[r, "d"] & \cdots \end{tikzcd} \tag{57}$$

gives another quasi-isomorphism of dga's.

An immediate consequence of this construction is Dolbeault's theorem:

$$H^{0,q}_{\bar\partial}(X, B) \cong \check{H}^q(X, \mathscr{B}). \tag{58}$$

We have done a little more than just prove this fact however. We have also given maps that induce this isomorphism and described how the natural product structures are also mapped.

We may treat sheaf cohomology in a similar way. We use a double complex given by

$$\tilde{E}_0^{p,q} = \check{C}^p(\mathfrak{U}, \mathscr{I}^q), \quad d = \delta + (-1)^p i_q \tag{59}$$

A quasi-isomorphism analogous to (57) again follows. Since the sheaves $\mathscr{I}^q$ are "flabby", one may also show that [39]

$$0 \longrightarrow \Gamma(\mathscr{I}^q) \xrightarrow{\rho} \check{C}^0(\mathfrak{U}, \mathscr{I}^q) \xrightarrow{\delta} \check{C}^1(\mathfrak{U}, \mathscr{I}^q) \xrightarrow{\delta} \check{C}^2(\mathfrak{U}, \mathscr{I}^q) \xrightarrow{\delta} \ldots \tag{60}$$

is exact. This gives rise to yet another quasi-isomorphism of dga's:

$$\begin{tikzcd} \cdots \arrow[r, "\Gamma(i_{n-2})"] & \Gamma(\mathscr{I}^{n-1}) \arrow[r, "\Gamma(i_{n-1})"] \arrow[d, "\rho"] & \Gamma(\mathscr{I}^{n}) \arrow[r, "\Gamma(i_{n})"] \arrow[d, "\rho"] & \Gamma(\mathscr{I}^{n+1}) \arrow[r, "\Gamma(i_{n+1})"] \arrow[d, "\rho"] & \cdots \\ \cdots \arrow[r, "d"] & \tilde{E}^{n-1} \arrow[r, "d"] & \tilde{E}^{n} \arrow[r, "d"] & \tilde{E}^{n+1} \arrow[r, "d"] & \cdots \end{tikzcd} \tag{61}$$

where

$$\tilde{E}^n = \bigoplus_{p+q=n} \tilde{E}_0^{p,q}, \tag{62}$$

Finally, note that the injective resolution (48) induces a quasi-isomorphism from (47) to the bottom complex of (61). Thus all of the complexes we have discussed are quasi-isomorphic to each other. By Lemma 1, all of the A_∞-algebras we obtain are A_∞-isomorphic to each other. In particular, combining this result with the discussion at the end of Section 3, we conclude that the superpotential of holomorphic Chern-Simons theory is independent of the metric up to field redefinitions, an expected property of the B-model.

One is therefore free to recast the formulation of the topological B-model into either Čech cohomology or sheaf cohomology. The idea that one may use sheaf cohomology leads inexorably to the appearance of the derived category $\mathbf{D}(X)$, as reviewed in [9]. So far we have restricted attention to a single D-brane that fills X. One extends this notion to any number of more general D-branes. The result is that a D-brane is a complex of coherent sheaves

$$\mathscr{E}^\bullet = \left(\cdots \xrightarrow{\mathsf{d}_{n-2}} \mathscr{E}^{n-1} \xrightarrow{\mathsf{d}_{n-1}} \mathscr{E}^{n} \xrightarrow{\mathsf{d}_{n}} \mathscr{E}^{n+1} \xrightarrow{d_{n+1}} \cdots \right). \tag{63}$$

For the analysis of open strings between $\mathscr{E}^\bullet$ and $\mathscr{F}^\bullet$ one replaces $\mathscr{B}$ in the above discussion by the sheaf $\mathscr{H}om(\mathscr{E}^m, \mathscr{F}^n)$. One needs to extend the notation to cope with these new complexes but this is an exercise only in bookkeeping and we will spare the reader of this. The Hilbert space of open strings from $\mathscr{E}^\bullet$ to $\mathscr{F}^\bullet$ is then given by "hyperext" groups:

$$\bigoplus_n \operatorname{Ext}^n(\mathscr{E}^\bullet, \mathscr{F}^\bullet). \tag{64}$$

Let us review exactly how to compute the A_∞ structure of the morphisms between objects in $\mathbf{D}(X)$. Each object in $\mathbf{D}(X)$ is quasi-isomorphic to a complex of injective sheaves. We may view this as an injective resolution of these objects. Without loss of generality therefore, we may assume that the D-branes are given as a complex of injective sheaves. Suppose, first, for simplicity, that we have only one D-brane $\mathscr{E}^\bullet$. The first row of (61) is then given by the complex with entries

$$\bigoplus_p \operatorname{Hom}(\mathscr{E}^p, \mathscr{E}^{p+n}). \tag{65}$$

If we denote an element of this group by $\sum_p f_{n,p}$, where $f_{n,p} : \mathscr{E}^p \to \mathscr{E}^{p+n}$, then the differential for this complex is given by

$$\mathfrak{d}_n f_{n,p} = \mathsf{d}_{p+n} \circ f_{n,p} - (-1)^n f_{p+1,n} \circ \mathsf{d}_p. \tag{66}$$

For several D-branes, we write $\mathscr{E}^\bullet = \mathscr{E}_1^\bullet \oplus \mathscr{E}_2^\bullet \oplus \ldots$. The spaces of Hom's then break up into direct sums and we may relabel everything in terms of morphisms between the different objects $\mathscr{E}_1^\bullet$, $\mathscr{E}_2^\bullet$, etc.

This complex, together with the obvious product structure given by composition, gives a dga. The cohomology of this complex gives the Hilbert spaces of the various open string states. The method of section 2 may then be used to compute the higher products of the resulting A_∞ category and thus we find the information required for the superpotential.

5 A Practical Method

In the last section we achieved our primary goal. We rephrased the question of how to compute the superpotential into a purely algebraic one. There is no need to know the metric on X. Having said that, the answer we obtained cannot really be viewed as a practical method of computing the higher products. This is because it required finding an injective resolution for each sheaf involved. While the existence of injective resolutions is guaranteed (see [39] for example), an explicit construction is not usually forthcoming.

Instead we should use Čech cohomology as follows. In general, there is a spectral sequence given by[3]

$$E_2^{p,q} = H^p(X, \mathscr{E}xt^q(\mathscr{E}, \mathscr{F})), \tag{67}$$

that converges to $\operatorname{Ext}^{p+q}(\mathscr{E}, \mathscr{F})$. If $\mathscr{E}$ is locally-free, then $\mathscr{E}xt^q(\mathscr{E}, \mathscr{F}) = 0$ for $q > 0$ and therefore

$$\begin{aligned} \operatorname{Ext}^n(\mathscr{E}, \mathscr{F}) &= H^n(X, \mathscr{H}om(\mathscr{E}, \mathscr{F})) \\ &= \check{H}^n(X, \mathscr{H}om(\mathscr{E}, \mathscr{F})). \end{aligned} \tag{68}$$

Now any coherent sheaf on a smooth X, has a locally-free resolution and so we are free to represent any object of $\mathbf{D}(X)$ by a complex of locally-free sheaves. Unlike the case of injectives representations, it is usually straightforward to compute a locally-free representation of a given object in $\mathbf{D}(X)$.

So we proceed as follows. Suppose, again, for simplicity of notation, that we have a single D-brane which is represented by a complex $\mathscr{E}^\bullet$ of locally-free sheaves. We have a complex with entries denoted:

$$\mathscr{H}om^q(\mathscr{E}^\bullet, \mathscr{E}^\bullet) = \bigoplus_m \mathscr{H}om(\mathscr{E}^m, \mathscr{E}^{m+q}), \tag{69}$$

and a differential $\mathfrak{d}_q$ given by (66). Now build a double complex with entries

$$\bigoplus_{p+q=n} \check{C}^p\left(\mathfrak{U}, \mathscr{H}om^q(\mathscr{E}^\bullet, \mathscr{E}^\bullet)\right), \tag{70}$$

of degree n, and differential $d = \delta + (-1)^p \mathfrak{d}_q$, where $\mathfrak{d}_q$ is given by (66).

[3]This "local to global" spectral sequence can be viewed as the Grothendieck spectral sequence (see [40] for example) applied to the composition of functors Γ and $\mathscr{H}om(\mathscr{E}, -)$.

There is a natural product, given by the Čech cup product combined with composition of maps in the $\mathscr{H}om$ sheaves. Suppose

$$\begin{aligned} \mathsf{a} &\in \check{C}^p\left(\mathfrak{U}, \mathscr{H}om^q(\mathscr{E}^\bullet, \mathscr{E}^\bullet)\right) \\ \mathsf{b} &\in \check{C}^r\left(\mathfrak{U}, \mathscr{H}om^s(\mathscr{E}^\bullet, \mathscr{E}^\bullet)\right) \end{aligned} \tag{71}$$

and let us denote the natural composition $\mathsf{a}\cdot\mathsf{b}$. This composition fails to satisfy the required Leibniz rule and instead we define a product

$$\mathsf{a} \star \mathsf{b} = (-1)^{qr}\mathsf{a}\cdot\mathsf{b}. \tag{72}$$

This new product gives us the structure of a dga.

By the same methods that were employed above, this dga is again quasi-isomorphic to all those considered in section 4. The presentation of the dga is actually perfectly practical to use, at least in relatively simple cases as we now demonstrate.

In order to compute Čech cohomology, we need an open cover of X that is sufficiently fine. That is, we need all the open sets, and all the intersections of the open sets, to have trivial sheaf cohomology. A sufficient condition for this is that the open sets and their intersections be *affine* [39]. A space is affine if it can be written as the solution of a set of algebraic equations in $\mathbb{C}^n$, for some n.

For example, consider the projective space $\mathbb{P}^n$ with homogeneous coordinates $[z_0, z_1, \ldots, z_n]$. The usual patches U_i, isomorphic to $\mathbb{C}^n$, are defined by $z_i \neq 0$. Let

$$U_{i_0 i_1 \ldots i_p} = U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_p}. \tag{73}$$

The space $U_{i_0,i_1\ldots i_p} \cong (C^*)^p \times \mathbb{C}^{n-p}$ is isomorphic to the affine variety defined by $z_{i_0} z_{i_1} \ldots z_{i_p} = 1$ in $\mathbb{C}^{n+1}$. Thus this cover is good enough for our purposes. Note that any algebraic variety defined within this $\mathbb{P}^n$ can also use this cover.

Before giving some examples, let us fix notation for Čech cochains. In our examples, our sheaves $\mathscr{F}$ will be vector bundles which have been trivialized over each U_i, so that sections of $\mathscr{F}$ over U_i can and will be identified with tuples of functions on U_i. As usual, when we change trivializations, we must multiply by an appropriate transition function.

For the higher cochains we will make a notational choice in describing elements of $\mathscr{F}(U_{i_0,i_1,\ldots,i_p})$ since many different trivializations are possible in general. Our choice will consistently be to choose the trivialization over U_{i_0}. So $(f)_{i_0,i_1,\ldots,i_p}$ denotes a section of $\mathscr{F}(U_{i_0,i_1,\ldots,i_p})$ over $U_{i_0,i_1,\ldots,i_p}$, expressed as a vector of functions using the given trivialization of $\mathscr{F}$ over U_{i_0}.

As a special case, if a 0-chain is a global section, i.e., a 0-cocycle, then we denote it simply by f, when f denotes its expression in the U_0 trivialization.

For example, for the sheaf $\mathscr{O}(n)$ on $\mathbb{P}^1$, $(f)_{01}$ is a 1-cochain given by f in terms of variables for U_0. It will therefore be given by $(f)_{01}.(z_0/z_1)^n$ in terms of variables in the patch U_1.

5.1 The conifold point of type $(-1,-1)$

For the first example consider a 3-brane (i.e., a point-like object in the compact directions) on a conifold point obtained by contracting a curve $C \cong \mathbb{P}^1$ with normal bundle $\mathscr{O}_C(-1) \oplus$

$\mathscr{O}_C(-1)$. As explained in [41], this 3-brane is marginally stable with respect to decay into $\mathscr{O}_C$ and $\mathscr{O}_C(-1)[1]$. Thus, if we considered N coincident 3-branes at this conifold point, we would have a $\mathrm{U}(N)\times\mathrm{U}(N)$ quiver gauge theory.

The superpotential for this case is known. It is computed in [10,11] by somewhat indirect means. This will provide a useful check for our method of computation. One may also regard our computation as a more rigorous proof of the result.

To produce a local model for this case, let X be the total space of the normal bundle $\mathscr{O}_C(-1)\oplus\mathscr{O}_C(-1)$. Thus we have bundle map $\pi : X \to C$. An affine open cover of X is then given by two patches: U_0, with coordinates (x, y_1, y_2); and U_1, with coordinates (w, z_1, z_2). The transition functions are obviously

$$\begin{aligned} w &= x^{-1} \\ z_1 &= xy_1 \\ z_2 &= xy_2 \end{aligned} \tag{74}$$

Now $\mathscr{O}_C$ is not a locally-free sheaf on X. Define $\mathscr{O}(1) = \pi^*\mathscr{O}_C(1)$. We then have an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{O}(2) \arrow[r, "\left(\begin{smallmatrix}-y_2\\ y_1\end{smallmatrix}\right)", "\left(\begin{smallmatrix}-z_2\\ z_1\end{smallmatrix}\right)"'] & \mathscr{O}(1)\oplus\mathscr{O}(1) \arrow[r, "(y_1\ y_2)", "(z_1\ z_2)"'] & \mathscr{O} \arrow[r] & \mathscr{O}_C \arrow[r] & 0, \end{tikzcd} \tag{75}$$

where we have given the explicit sheaf maps in both patches. This provides the locally-free resolution of $\mathscr{O}_C$, and thus $\mathscr{O}_C(-1)[1]$ too by tensoring the resolution by $\mathscr{O}(-1)$ and shifting one place to the left.

$\mathrm{Ext}^1(\mathscr{O}_C(-1)[1], \mathscr{O}_C)$ and $\mathrm{Ext}^1(\mathscr{O}_C, \mathscr{O}_C(-1)[1])$ are both isomorphic to $\mathbb{C}^2$. Thus we have a quiver:

$$\begin{tikzcd}[column sep=huge] \mathscr{O}_C(-1)[1]\ \circ \arrow[r, bend left=40, "\mathsf{a}"] \arrow[r, bend left=12, "\mathsf{b}"] & \circ\ \mathscr{O}_C \arrow[l, bend left=12, "\mathsf{c}"] \arrow[l, bend left=40, "\mathsf{d}"] \end{tikzcd} \tag{76}$$

Open strings correspond to maps which are d-closed. In turns out that in this example we may represent all the required d-closed maps by maps which are both δ-closed and $\mathfrak{d}$-closed as we now see explicitly. The classes in $\mathrm{Ext}^1(\mathscr{O}_C(-1)[1], \mathscr{O}_C)$ are represented by elements of $\check{C}^0(\mathfrak{U}, \mathscr{H}\!om^1(\mathscr{O}_C(-1)[1], \mathscr{O}_C))$ as follows. Using the notation described above, let one generator of this group, denoted a, be represented by

$$\begin{tikzcd} \mathscr{O}(1) \arrow[r, "\left(\begin{smallmatrix}-y_2\\ y_1\end{smallmatrix}\right)"] \arrow[d, "1"] & \mathscr{O}\oplus\mathscr{O} \arrow[r, "(y_1\ y_2)"] \arrow[d, "-\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)"] & \mathscr{O}(-1) \arrow[d, "1"] \\ \mathscr{O}(2) \arrow[r, "\left(\begin{smallmatrix}-y_2\\ y_1\end{smallmatrix}\right)"] & \mathscr{O}(1)\oplus\mathscr{O}(1) \arrow[r, "(y_1\ y_2)"] & \mathscr{O} \end{tikzcd} \tag{77}$$

and b by

$$
\begin{tikzcd}
\mathscr{O}(1) \arrow[r, "\left(\begin{smallmatrix} -y_2 \\ y_1 \end{smallmatrix}\right)"] \arrow[d, "x"] & \mathscr{O} \oplus \mathscr{O} \arrow[r, "\left(\begin{smallmatrix} y_1 & y_2 \end{smallmatrix}\right)"] \arrow[d, "-\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)"] & \mathscr{O}(-1) \arrow[d, "x"] \\
\mathscr{O}(2) \arrow[r, "\left(\begin{smallmatrix} -y_2 \\ y_1 \end{smallmatrix}\right)"] & \mathscr{O}(1) \oplus \mathscr{O}(1) \arrow[r, "\left(\begin{smallmatrix} y_1 & y_2 \end{smallmatrix}\right)"] & \mathscr{O}.
\end{tikzcd}
\tag{78}
$$

Next, the two generators of $\mathrm{Ext}^1(\mathscr{O}_C, \mathscr{O}_C(-1)[1])$ can be represented by elements of $\check{C}^1(\mathfrak{U}, \mathscr{H}om^0(\mathscr{O}_C, \mathscr{O}_C(-1)[1])$. Let c be represented by

$$
\begin{tikzcd}
& \mathscr{O}(2) \arrow[r, "\left(\begin{smallmatrix} -y_2 \\ y_1 \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 0 \\ -\frac{1}{x} \end{smallmatrix}\right)_{01}"] & \mathscr{O}(1) \oplus \mathscr{O}(1) \arrow[r, "\left(\begin{smallmatrix} y_1 & y_2 \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} \frac{1}{x} & 0 \end{smallmatrix}\right)_{01}"] & \mathscr{O} \\
\mathscr{O}(1) \arrow[r, "\left(\begin{smallmatrix} -y_2 \\ y_1 \end{smallmatrix}\right)"] & \mathscr{O} \oplus \mathscr{O} \arrow[r, "\left(\begin{smallmatrix} y_1 & y_2 \end{smallmatrix}\right)"] & \mathscr{O}(-1) &
\end{tikzcd}
\tag{79}
$$

and d by

$$
\begin{tikzcd}
& \mathscr{O}(2) \arrow[r, "\left(\begin{smallmatrix} -y_2 \\ y_1 \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} \frac{1}{x} \\ 0 \end{smallmatrix}\right)_{01}"] & \mathscr{O}(1) \oplus \mathscr{O}(1) \arrow[r, "\left(\begin{smallmatrix} y_1 & y_2 \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 0 & \frac{1}{x} \end{smallmatrix}\right)_{01}"] & \mathscr{O} \\
\mathscr{O}(1) \arrow[r, "\left(\begin{smallmatrix} -y_2 \\ y_1 \end{smallmatrix}\right)"] & \mathscr{O} \oplus \mathscr{O} \arrow[r, "\left(\begin{smallmatrix} y_1 & y_2 \end{smallmatrix}\right)"] & \mathscr{O}(-1) &
\end{tikzcd}
\tag{80}
$$

Finally, the generator of $\mathrm{Ext}^3(\mathscr{O}_C(-1)[1], \mathscr{O}_C(-1)[1])$ can be represented by a 1-cochain in $\check{C}^1(\mathfrak{U}, \mathscr{H}om^2(\mathscr{O}_C(-1)[1], \mathscr{O}_C(-1)[1]))$:

$$
\begin{tikzcd}
& & \mathscr{O}(1) \arrow[r, "\left(\begin{smallmatrix} -y_2 \\ y_1 \end{smallmatrix}\right)"] \arrow[d, "(\frac{1}{x})_{01}"] & \mathscr{O} \oplus \mathscr{O} \arrow[r, "\left(\begin{smallmatrix} y_1 & y_2 \end{smallmatrix}\right)"] & \mathscr{O}(-1) \\
\mathscr{O}(1) \arrow[r, "\left(\begin{smallmatrix} -y_2 \\ y_1 \end{smallmatrix}\right)"] & \mathscr{O} \oplus \mathscr{O} \arrow[r, "\left(\begin{smallmatrix} y_1 & y_2 \end{smallmatrix}\right)"] & \mathscr{O}(-1) & &
\end{tikzcd}
\tag{81}
$$

The composition $\mathsf{c} \star \mathsf{a}$ gives a map

$$\begin{tikzcd} & \mathscr{O}(1) \arrow[r, "\binom{-y_2}{y_1}"] \arrow[d, "\binom{0}{-\frac{1}{x}}_{01}"] & \mathscr{O} \oplus \mathscr{O} \arrow[r, "(y_1\ y_2)"] \arrow[d, "\left(-\frac{1}{x}\ 0\right)_{01}"] & \mathscr{O}(-1) \\ \mathscr{O}(1) \arrow[r, "\binom{-y_2}{y_1}"] & \mathscr{O} \oplus \mathscr{O} \arrow[r, "(y_1\ y_2)"] & \mathscr{O}(-1) & \end{tikzcd} \tag{82}$$

This is exact. To be precise, $\mathsf{c} \star \mathsf{a}$ is a Čech coboundary of the map which is zero in patch 0 and in patch 1 given by the chain map

$$\begin{tikzcd} & \mathscr{O}(1) \arrow[r, "\binom{-y_2}{y_1}"] \arrow[d, "\binom{0}{-1}_{1}"] & \mathscr{O} \oplus \mathscr{O} \arrow[r, "(y_1\ y_2)"] \arrow[d, "(-1\ 0)_{1}"] & \mathscr{O}(-1) \\ \mathscr{O}(1) \arrow[r, "\binom{-y_2}{y_1}"] & \mathscr{O} \oplus \mathscr{O} \arrow[r, "(y_1\ y_2)"] & \mathscr{O}(-1) & \end{tikzcd} \tag{83}$$

Thus, from (15), $m_2(\mathsf{c},\mathsf{a}) = 0$ and $f_2(\mathsf{c},\mathsf{a})$ is given by minus (83).

Now compose this with b to form $\mathsf{b} \star f_2(\mathsf{c},\mathsf{a})$ given by the Čech 0-chain

$$\begin{tikzcd} & \mathscr{O}(1) \arrow[r, "\binom{-y_2}{y_1}"] \arrow[d, "\binom{0}{-1}_{1}"] & \mathscr{O} \oplus \mathscr{O} \arrow[r, "(y_1\ y_2)"] \arrow[d, "(1\ 0)_{1}"] & \mathscr{O}(-1) \\ \mathscr{O}(2) \arrow[r, "\binom{-y_2}{y_1}"] & \mathscr{O}(1) \oplus \mathscr{O}(1) \arrow[r, "(y_1\ y_2)"] & \mathscr{O} & \end{tikzcd} \tag{84}$$

This corresponds to one of the terms needed to compute $m_3(\mathsf{b},\mathsf{c},\mathsf{a})$.

Similarly, a computation for $f_2(\mathsf{b},\mathsf{c}) \star \mathsf{a}$ yields

$$\begin{tikzcd} & \mathscr{O}(1) \arrow[r, "\binom{-y_2}{y_1}"] \arrow[d, "\binom{0}{-1}_{0}"] & \mathscr{O} \oplus \mathscr{O} \arrow[r, "(y_1\ y_2)"] \arrow[d, "(1\ 0)_{0}"] & \mathscr{O}(-1) \\ \mathscr{O}(2) \arrow[r, "\binom{-y_2}{y_1}"] & \mathscr{O}(1) \oplus \mathscr{O}(1) \arrow[r, "(y_1\ y_2)"] & \mathscr{O} & \end{tikzcd} \tag{85}$$

Remembering the rule (2), from (16) we see that $m_3(\mathsf{b},\mathsf{c},\mathsf{a})$ is equal to $-\mathsf{b}\star f_2(\mathsf{c},\mathsf{a}) - f_2(\mathsf{b},\mathsf{c})\star\mathsf{a}$ and is thus given by the following globally defined map which represents a generator of $\mathrm{Ext}^2(\mathscr{O}_C(-1)[1],\mathscr{O}_C)$:

$$\begin{tikzcd} & \mathscr{O}(1) \arrow[r, "\left(\begin{smallmatrix}-y_2\\ y_1\end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix}0\\ 1\end{smallmatrix}\right)"] & \mathscr{O}\oplus\mathscr{O} \arrow[r, "(y_1\ y_2)"] \arrow[d, "(-1\ 0)"] & \mathscr{O}(-1) \\ \mathscr{O}(2) \arrow[r, "\left(\begin{smallmatrix}-y_2\\ y_1\end{smallmatrix}\right)"] & \mathscr{O}(1)\oplus\mathscr{O}(1) \arrow[r, "(y_1\ y_2)"] & \mathscr{O} & \end{tikzcd} \tag{86}$$

When composed with d this gives the Ext^3 of (81) but when composed with c it gives zero. Thus $m_3(\mathsf{b},\mathsf{c},\mathsf{a})$ is Serre dual to d. Denoting by A the $N=1$ superfield dual to a etc., we thus have a term in the superpotential equal to $\mathrm{Tr}(BCAD)$.

Composing the other way to find $m_3(\mathsf{a},\mathsf{c},\mathsf{b})$ gives a similar result except for a sign. There are no other higher products and so the total is, in agreement with [10]:

$$W = \mathrm{Tr}(BCAD - ACBD).$$

One is also free to do nonlinear field redefinitions as discussed at the end of section 3.

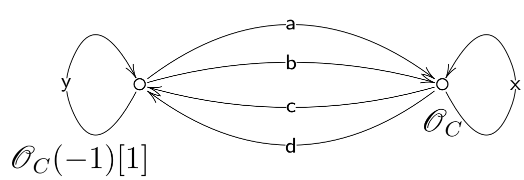
5.2 A $\mathbb{P}^1$ with higher obstructions.

Our next example is a conifold-like point associated with an obstructed $\mathbb{P}^1$ with normal bundle $\mathscr{O}\oplus\mathscr{O}(-2)$. An example of such a $\mathbb{P}^1$ can be given explicitly in patches using the transition functions

$$\begin{aligned} w &= x^{-1} \\ z_1 &= x^2 y_1 + x y_2^n \\ z_2 &= y_2 \end{aligned} \tag{87}$$

with $n \geq 2$ (the $n=1$ case can be identified with the resolved conifold after a change of variables).

The quiver for a decay of a 3-brane into $\mathscr{O}_C$ and $\mathscr{O}_C(-1)[1]$ in this case is given by



(88)

A locally-free resolution of $\mathscr{O}_C$ is given by

$$\begin{tikzcd} \mathscr{O} \arrow[r, "\begin{pmatrix} y_2 \\ -1 \\ x \end{pmatrix}"] & \begin{matrix} \mathscr{O} \\ \oplus \\ \mathscr{O}(1) \\ \oplus \\ \mathscr{O}(1) \end{matrix} \arrow[r, "\begin{pmatrix} 1 & y_2 & 0 \\ -x & 0 & y_2 \\ -y_2^{n-1} & -s & -y_1 \end{pmatrix}"] & \begin{matrix} \mathscr{O}(1) \\ \oplus \\ \mathscr{O}(1) \\ \oplus \\ \mathscr{O} \end{matrix} \arrow[r, "(\, s \ y_1 \ y_2 \,)"] & \mathscr{O} \arrow[r] & \mathscr{O}_C. \end{tikzcd} \tag{89}$$

where $s = xy_1 + y_2^n$.

In constructing this resolution, the bundles $\mathscr{O}(n)$ which appear were chosen so that all maps appearing in the resolution remain holomorphic in the U_1 after changing coordinates and multiplying by the transition function x^{-n} of $\mathscr{O}(n)$. For example, s is given as a section of $\mathscr{O}(-1)$. In U_1 coordinates, this becomes $z_1 = xs$ which is holomorphic.

Note that the sections s, y_1, y_2 have been chosen to generate the ideal of all functions vanishing on C in both patches. In U_0, the sections y_1 and y_2 already suffice to generate the ideal. In U_1, these sections become $z_1, wz_1 - z_2^n, z_2$ respectively, and now z_1 and z_2 already suffice. Note in particular that it was necessary to include the section s, as y_1, y_2 would not have sufficed: in U_1 these get identified with $wz_1 - z_2^n, z_2$ which fail to generate the ideal of the curve at $(w, z_1, z_2) = (0, 0, 0)$.

Define x to be the following generator of $\mathrm{Ext}^1(\mathscr{O}_C, \mathscr{O}_C) \cong \mathbb{C}$:

$$\begin{tikzcd} & \mathscr{O} \arrow[r, "\begin{pmatrix} y_2 \\ -1 \\ x \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}"] & \begin{matrix} \mathscr{O} \\ \oplus \\ \mathscr{O}(1) \\ \oplus \\ \mathscr{O}(1) \end{matrix} \arrow[r, "\begin{pmatrix} 1 & y_2 & 0 \\ -x & 0 & y_2 \\ -y_2^{n-1} & -s & -y_1 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ y_2^{n-2} & 0 & 0 \end{pmatrix}"] & \begin{matrix} \mathscr{O}(1) \\ \oplus \\ \mathscr{O}(1) \\ \oplus \\ \mathscr{O} \end{matrix} \arrow[r, "(\, s \ y_1 \ y_2 \,)"] \arrow[d, "(0\ 0\ 1)"] & \mathscr{O} \\ \mathscr{O} \arrow[r, "\begin{pmatrix} y_2 \\ -1 \\ x \end{pmatrix}"] & \begin{matrix} \mathscr{O} \\ \oplus \\ \mathscr{O}(1) \\ \oplus \\ \mathscr{O}(1) \end{matrix} \arrow[r, "\begin{pmatrix} 1 & y_2 & 0 \\ -x & 0 & y_2 \\ -y_2^{n-1} & -s & -y_1 \end{pmatrix}"] & \begin{matrix} \mathscr{O}(1) \\ \oplus \\ \mathscr{O}(1) \\ \oplus \\ \mathscr{O} \end{matrix} \arrow[r, "(\, s \ y_1 \ y_2 \,)"] & \mathscr{O} & \end{tikzcd} \tag{90}$$

From now on, for brevity, let us refer to the sheaves in the locally-free resolution (89) as $\mathscr{F}_i$. $\mathrm{Ext}^3(\mathscr{O}_C, \mathscr{O}_C)$ is represented by the 0-cochain:

$$\begin{tikzcd} & & & \mathscr{F}_3 \arrow[r] \arrow[d, "1"] & \mathscr{F}_2 \arrow[r] & \mathscr{F}_1 \arrow[r] & \mathscr{F}_0 \\ \mathscr{F}_3 \arrow[r] & \mathscr{F}_2 \arrow[r] & \mathscr{F}_1 \arrow[r] & \mathscr{F}_0 & & & \end{tikzcd} \tag{91}$$

or, equivalently, by the 1-cochain:

$$
\begin{tikzcd}
 & & \mathscr{F}_3 \arrow[r] \arrow[d, "0"] & \mathscr{F}_2 \arrow[r] \arrow[d, "\begin{pmatrix} 0 & 1 & \frac{1}{x} \end{pmatrix}_{01}"] & \mathscr{F}_1 \arrow[r] & \mathscr{F}_0 \\
\mathscr{F}_3 \arrow[r] & \mathscr{F}_2 \arrow[r] & \mathscr{F}_1 \arrow[r] & \mathscr{F}_0
\end{tikzcd}
\tag{92}
$$

These two choices differ by a d-boundary. We will use the representative (91) to describe the A_∞-algebra via Kadeishvili's theorem.

We compute $\mathsf{x} \star \mathsf{x}$ to be J_{n-2}, where $\mathsf{J}_p \in \mathrm{Ext}^2(\mathscr{O}_C, \mathscr{O}_C)$ is defined as

$$
\begin{tikzcd}
 & & \mathscr{F}_3 \arrow[r] \arrow[d, "\begin{pmatrix} 0 \\ 0 \\ y_2^p \end{pmatrix}"] & \mathscr{F}_2 \arrow[r] \arrow[d, "\begin{pmatrix} y_2^p & 0 & 0 \end{pmatrix}"] & \mathscr{F}_1 \arrow[r] & \mathscr{F}_0 \\
\mathscr{F}_3 \arrow[r] & \mathscr{F}_2 \arrow[r] & \mathscr{F}_1 \arrow[r] & \mathscr{F}_0
\end{tikzcd}
\tag{93}
$$

But, if $p \geq 1$ then

$$
\mathsf{J}_p = d\mathsf{K}_{p-1}, \tag{94}
$$

where K_p is given by

$$
\begin{tikzcd}
 & \mathscr{F}_3 \arrow[r] \arrow[d, "\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}"] & \mathscr{F}_2 \arrow[r] \arrow[d, "\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ y_2^p & 0 & 0 \end{pmatrix}"] & \mathscr{F}_1 \arrow[r] \arrow[d, "\begin{pmatrix} 0 & 0 & 0 \end{pmatrix}"] & \mathscr{F}_0 \\
\mathscr{F}_3 \arrow[r] & \mathscr{F}_2 \arrow[r] & \mathscr{F}_1 \arrow[r] & \mathscr{F}_0
\end{tikzcd}
\tag{95}
$$

It is now easy to see that

$$
\mathsf{J}_p = \mathsf{x} \star \mathsf{K}_p + \mathsf{K}_p \star \mathsf{x}. \tag{96}
$$

Applying (12) and using the fact that $\mathsf{K}_i \star \mathsf{K}_j = 0$, it follows that we can choose

$$
\left.\begin{aligned}
f_k(\mathsf{x}, \mathsf{x}, \dots, \mathsf{x}) &= (-1)^{\frac{k(k-1)}{2}} \mathsf{K}_{n-k-1} \\
m_k(\mathsf{x}, \mathsf{x}, \dots, \mathsf{x}) &= 0
\end{aligned}\right\} \quad \text{for } 2 \leq k < n. \tag{97}
$$

and

$$
m_n(\mathsf{x}, \mathsf{x}, \dots, \mathsf{x}) = -(-1)^{\frac{n(n-1)}{2}} \mathsf{J}_0. \tag{98}
$$

But J_0 composed with x is the generator of Ext^3 given in (91) so we have a term in the superpotential equal to $-(-1)^{\frac{n(n-1)}{2}} X^{n+1}$. Similarly we obtain a contribution $-(-1)^{\frac{n(n-1)}{2}} Y^{n+1}$ to the superpotential.

The next few arrows in (88) are given by:

$$\mathsf{a} = \begin{tikzcd} \mathscr{F}_3(-1) \arrow[r] \arrow[d, "-1"] & \mathscr{F}_2(-1) \arrow[r] \arrow[d, "\mathbf{1}"] & \mathscr{F}_1(-1) \arrow[r] \arrow[d, "-\mathbf{1}"] & \mathscr{F}_0(-1) \arrow[d, "1"] \\ \mathscr{F}_3 \arrow[r] & \mathscr{F}_2 \arrow[r] & \mathscr{F}_1 \arrow[r] & \mathscr{F}_0 \end{tikzcd} \tag{99}$$

$$\mathsf{b} = \begin{tikzcd} \mathscr{F}_3(-1) \arrow[r] \arrow[d, "-x"] & \mathscr{F}_2(-1) \arrow[r] \arrow[d, "x.\mathbf{1}"] & \mathscr{F}_1(-1) \arrow[r] \arrow[d, "-x.\mathbf{1}"] & \mathscr{F}_0(-1) \arrow[d, "x"] \\ \mathscr{F}_3 \arrow[r] & \mathscr{F}_2 \arrow[r] & \mathscr{F}_1 \arrow[r] & \mathscr{F}_0 \end{tikzcd} \tag{100}$$

$$\mathsf{c} = \begin{tikzcd} & \mathscr{F}_3 \arrow[r] \arrow[d, "0"] & \mathscr{F}_2 \arrow[r] \arrow[d, "{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & \frac{1}{x} \end{pmatrix}_{01}}"] & \mathscr{F}_1 \arrow[r] \arrow[d, "{\begin{pmatrix} 1 & \frac{1}{x} & 0 \end{pmatrix}_{01}}"] & \mathscr{F}_0 \\ \mathscr{F}_3(-1) \arrow[r] & \mathscr{F}_2(-1) \arrow[r] & \mathscr{F}_1(-1) \arrow[r] & \mathscr{F}_0(-1) & \end{tikzcd} \tag{101}$$

A new feature appears when we try to write down the final map d. Unlike the above cases we cannot use a single map with $\delta\mathsf{d} = \mathfrak{d}\mathsf{d} = 0$. Instead we need to write d as a sum $f + h$, where f is a class in $\check{C}^1(\mathfrak{U}, \mathscr{H}om^0(\mathscr{O}, \mathscr{O}(-1)[1]))$:

$$\begin{tikzcd} & \mathscr{F}_3 \arrow[r] \arrow[d, "0"] & \mathscr{F}_2 \arrow[r] \arrow[d, "{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{1}{x} & 0 \end{pmatrix}_{01}}"] & \mathscr{F}_1 \arrow[r] \arrow[d, "{\begin{pmatrix} -\frac{1}{x} & 0 & 0 \end{pmatrix}_{01}}"] & \mathscr{F}_0 \\ \mathscr{F}_3(-1) \arrow[r] & \mathscr{F}_2(-1) \arrow[r] & \mathscr{F}_1(-1) \arrow[r] & \mathscr{F}_0(-1) & \end{tikzcd} \tag{102}$$

and h is a class in $\check{C}^0(\mathfrak{U}, \mathscr{H}om^1(\mathscr{O}, \mathscr{O}(-1)[1]))$:

$$\begin{tikzcd} & & \mathscr{F}_3 \arrow[r] \arrow[d, "{\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}_{1}}"] & \mathscr{F}_2 \arrow[r] \arrow[d, "{\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}_{1}}"] & \mathscr{F}_1 \arrow[r] & \mathscr{F}_0 \\ \mathscr{F}_3(-1) \arrow[r] & \mathscr{F}_2(-1) \arrow[r] & \mathscr{F}_1(-1) \arrow[r] & \mathscr{F}_0(-1) & & \end{tikzcd} \tag{103}$$

Then $d\mathsf{d} = -\mathfrak{d}f + \delta h = 0$ as required.

A straight-forward computation, whose details we omit, then yields

$$W = \mathrm{Tr}\Big(-(-1)^{\frac{n(n-1)}{2}} X^{n+1} - (-1)^{\frac{n(n-1)}{2}} Y^{n+1} - XAC + XBD - YCA + YDB\Big) \tag{104}$$

in agreement with [12] for example.

5.3 A new example of type $(1, -3)$

Here we consider a 5-brane wrapping a $\mathbb{P}^1$ locally given by

$$\begin{aligned} w &= x^{-1} \\ z_1 &= x^3 y_1 + y_2^2 \\ z_2 &= x^{-1} y_2 \end{aligned} \tag{105}$$

This curve cannot be contracted and so we do not consider 3-brane decay in this case. There are 2 massless open strings beginning and ending on this $\mathbb{P}^1$ and the moduli space is again obstructed as we see below. It already follows from [42] that the moduli space can be defined as the critical point locus of a superpotential-like function XY^2, but no claim was made there that this coincides with the physical superpotential. Our computations will show that this is indeed the physical superpotential.

The equation $w^2 z_1 - z_2^2 = x y_1$ shows that y_1 can be identified with a global section of $\mathscr{O}(-1)$. Similarly, the last equation in (105) shows that y_2 can be identified with a section of $\mathscr{O}(1)$.

A resolution of $\mathscr{O}_C$ yields the following complex of locally-free sheaves representing the D-brane:

$$\mathscr{O}(-3) \xrightarrow{\begin{pmatrix} y_2 \\ -x^3 \\ -1 \end{pmatrix}} \begin{matrix} \mathscr{O}(-2) \\ \oplus \\ \mathscr{O} \\ \oplus \\ \mathscr{O}(-1) \end{matrix} \xrightarrow{\begin{pmatrix} x^3 & y_2 & 0 \\ y_2 & -y_1 & z_1 \\ -1 & 0 & -y_2 \end{pmatrix}} \begin{matrix} \mathscr{O}(1) \\ \oplus \\ \mathscr{O}(-1) \\ \oplus \\ \mathscr{O} \end{matrix} \xrightarrow{\begin{pmatrix} y_1 & y_2 & z_1 \end{pmatrix}} \mathscr{O}\,. \tag{106}$$

In (106), we have used z_1 as an abbreviation for its expression $x^3 y_1 + y_2^2$ in the U_0 patch.

Define x and y to be the following generators of $\mathrm{Ext}^1(\mathscr{O}_C, \mathscr{O}_C) \cong \mathbb{C}$:

$$\mathsf{x} = \begin{tikzcd} & \mathscr{F}_3 \arrow[r] \arrow[d, "\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}"] & \mathscr{F}_2 \arrow[r] \arrow[d, "\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}"] & \mathscr{F}_1 \arrow[r] \arrow[d, "\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}"] & \mathscr{F}_0 \\ \mathscr{F}_3 \arrow[r] & \mathscr{F}_2 \arrow[r] & \mathscr{F}_1 \arrow[r] & \mathscr{F}_0 & \end{tikzcd} \tag{107}$$

$$\mathsf{y} = \begin{tikzcd} & \mathscr{F}_3 \arrow[r] \arrow[d, "\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}"] & \mathscr{F}_2 \arrow[r] \arrow[d, "\begin{pmatrix} 0 & x & 0 \\ -x & 0 & 0 \\ 0 & 0 & -x \end{pmatrix}"] & \mathscr{F}_1 \arrow[r] \arrow[d, "\begin{pmatrix} 0 & x & 0 \end{pmatrix}"] & \mathscr{F}_0 \\ \mathscr{F}_3 \arrow[r] & \mathscr{F}_2 \arrow[r] & \mathscr{F}_1 \arrow[r] & \mathscr{F}_0 & \end{tikzcd} \tag{108}$$

In the U_0 patch, we can simply write $\mathsf{y} = x\mathsf{x}$. Note that the entries of the above matrices remain holomorphic in the U_1 patch, and prevent us from multiplying the entries by any

higher powers of x. We compute $\mathsf{x}\star\mathsf{x}$ to be

$$\begin{tikzcd}
 & & \mathscr{F}_3 \arrow[r] \arrow[d, "\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}"] & \mathscr{F}_2 \arrow[r] \arrow[d, "(-1\ 0\ 0)"] & \mathscr{F}_1 \arrow[r] & \mathscr{F}_0 \\
\mathscr{F}_3 \arrow[r] & \mathscr{F}_2 \arrow[r] & \mathscr{F}_1 \arrow[r] & \mathscr{F}_0 & &
\end{tikzcd} \tag{109}$$

From this, we compute immediately that $\mathsf{y}\star\mathsf{x} = x(\mathsf{x}\star\mathsf{x})$ and $\mathsf{y}\star\mathsf{y} = x^2(\mathsf{x}\star\mathsf{x})$.

We now note that $\mathsf{x}\star\mathsf{x}$ is exact, given by $\mathfrak{d}$ applied to

$$\begin{tikzcd}
 & \mathscr{F}_3 \arrow[r] \arrow[d, "\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}"] & \mathscr{F}_2 \arrow[r] \arrow[d, "\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}"] & \mathscr{F}_1 \arrow[r] \arrow[d, "(0\ 0\ 1)"] & \mathscr{F}_0 \\
\mathscr{F}_3 \arrow[r] & \mathscr{F}_2 \arrow[r] & \mathscr{F}_1 \arrow[r] & \mathscr{F}_0 &
\end{tikzcd} \tag{110}$$

Since the nonzero entries of (110) are sections of $\mathscr{O}$, they are constants hence holomorphic in the U_1 patch as well. This also explains that we can't simply multiply (110) by x or x^2 to conclude exactness of $\mathsf{y}\star\mathsf{x}$, and $\mathsf{y}\star\mathsf{y}$, as these are not holomorphic in the U_1 patch. In fact, it can be checked that $\mathsf{y}\star\mathsf{x}$ and $\mathsf{y}\star\mathsf{y}$ generate $\mathrm{Ext}^2(\mathscr{O}_C, \mathscr{O}_C)$.

Next, we compute $\mathsf{x}\star\mathsf{x}\star\mathsf{x}$ to be

$$\begin{tikzcd}
 & & & \mathscr{F}_3 \arrow[r] \arrow[d, "-1"] & \mathscr{F}_2 \arrow[r] & \mathscr{F}_1 \arrow[r] & \mathscr{F}_0 \\
\mathscr{F}_3 \arrow[r] & \mathscr{F}_2 \arrow[r] & \mathscr{F}_1 \arrow[r] & \mathscr{F}_0 & & &
\end{tikzcd} \tag{111}$$

We immediately compute

$$\mathsf{y}\star\mathsf{x}\star\mathsf{x} = x\,(\mathsf{x}\star\mathsf{x}\star\mathsf{x})\,,\ \ \mathsf{y}\star\mathsf{y}\star\mathsf{x} = x^2\,(\mathsf{x}\star\mathsf{x}\star\mathsf{x})\,,\ \ \mathsf{y}\star\mathsf{y}\star\mathsf{y} = x^3\,(\mathsf{x}\star\mathsf{x}\star\mathsf{x})\,. \tag{112}$$

The $\mathfrak{d}$-exactness of $\mathsf{x}\star\mathsf{x}$ and $\mathfrak{d}$-closedness of x and y imply that $\mathsf{x}\star\mathsf{x}\star\mathsf{y}$ and $\mathsf{x}\star\mathsf{x}\star\mathsf{x}$ are $\mathfrak{d}$-exact. Note that $\mathsf{y}\star\mathsf{y}\star\mathsf{y}$ is $\mathfrak{d}$ of

$$\begin{tikzcd}
 & & \mathscr{F}_3 \arrow[r] \arrow[d, "\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}"] & \mathscr{F}_2 \arrow[r] \arrow[d, "(0\ -1\ 0)"] & \mathscr{F}_1 \arrow[r] & \mathscr{F}_0 \\
\mathscr{F}_3 \arrow[r] & \mathscr{F}_2 \arrow[r] & \mathscr{F}_1 \arrow[r] & \mathscr{F}_0 & &
\end{tikzcd} \tag{113}$$

It can be shown that $\mathsf{y}\star\mathsf{y}\star\mathsf{x}$ is not exact and generates $\mathrm{Ext}^3(\mathscr{O}_C, \mathscr{O}_C)$.

This shows that XY^2 is the only cubic term in the superpotential. It is not hard to show inductively that all $m_k = f_k = 0$ for $k > 2$. Therefore, we have no higher terms in the superpotential and so

$$W = \mathrm{Tr}(XY^2), \tag{114}$$

as might have been expected from [42].

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